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Common Fixed Point Theorem for Two Self Maps with ω – Distance in a Cone Metric Space

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Abstract

In this paper , we extend results of Sami Ullah Khan and Arjamand Bano [9] and prove some common fixed point theorem for a pair of weakly compatible mappings in cone metric spaces using ω – distance on X without using normality in cone metric space. Also we extend the results of H.Lakzian and F.Arabyani [7] and 'infimum' of a set is used in H.Lakzian and F.Arabyani [7] which may not be meaningful in the context of a cone metric space.we successfully avoided this difficulty in this paper by obtaining suitable modifications.

Mathematics Subjects Classification : 54H25 , 47H10

Key Words : Cone Metric Space, ω – Distance, Common Fixed Point, Compatible Maps

1 Introduction

Fixed point theory plays an important role in functional analysis, approximation theory, differential equations and applications such as boundary value problems etc. The concept of a metric space was introduced in 1906 by M. Frechet [2]. It furnishes the common idealization of a large number of mathematical, physical and other scientific constructs in which the notion of a "distance" appears. The objects under consideration may be most varied. They may be points, functions, sets and even the subjective experiences of sensations. What matters is the possibility of associating a non negative real number with each ordered pair of elements of certain set, and that the number associated with pairs and triples of such elements satisfies certain conditions.

Haung and Zhang [3] generalized the concept of metric Space, replacing the set of real numbers by an ordered Banach space, and obtained some fixed point theorems for mappings satisfying different contractive conditions. The metric space with w-distance was introduced by O. Kada et al [6]. These two concepts were combined together and Cone metric space with w– distance was introduced by H.Lakzian and F.Arabyani [7], and a fixed point theorem was proved. Abbas and Jungck [1] proved some common fixed point theorems for weakly compatible mappings in the setting of cone metric space. D. IIić and Rakoćević [4], Rezapour and Hamlbarani [8] also proved some common fixed point theorems on cone metric spaces. Our objective is to extend these concepts together to establish a common fixed point theorem for a pair of weakly compatible mappings in cone metric space using w-distance. Consequently, we improve and generalize various results existing in the literature.

In this paper , we extend results of Sami Ullah Khan and Arjamand Bano [9] and prove some common fixed point theorem for a pair of weakly compatible mappings in cone metric spaces using ω – distance on X without using normality in cone metric space. Also we extend the results of H.Lakzian and F.Arabyani [7] and 'infimum' of a set is used in H.Lakzian and F.Arabyani [7] which may not be meaningful in the context of a cone metric space.we successfully avoided this difficulty in this paper by obtaining suitable modifications.

2 Preliminaries

2.1 Definition: (L.G. Haung and X. Zhang [3])

Let E be a real Branch Space and P a subset of E. The set P is called a cone if

1. P is closed, non-empty and $P \neq \{0\}$;

2. $a, b \in R$, $a, b \geq 0$, $x, y \in P$ then $ax + by \in P$;

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3. P ∩ (-P) = {0}

For a given cone P of E ,we define a partial ordering $\leq$ on E with respect to P by

$x \leq y$ if and only if $y - x \in$ P. We write $x<y$ to indicate that $x \leq y$ but $x \neq y$, while $x<<y$ stands

for $y - x \in$ Int P, where Int P denotes the interior of P.

2.2 Definition: (L.G. Haung and X. Zhang [3])

Let E be a real Banach space and P be a cone of E . The cone P is called normal if

there is a number K > 0 such that for all $x, y \in$ E, $0 \leq x \leq y$ implies

$\| x\| \leq K\| y\|$

The least positive number K satisfying the above inequality is called the normal constant of P.

In the following, we always suppose that E is a real Banach space, P is a cone in E

and E is endowed with the partial ordering induced by P.

2.3 Definition: (L.G. Haung and X. Zhang [3])

Let X be a non-empty set. Suppose that the mapping d: X x X → P satisfies:

a. 0 < d (x, y) for all x, y∈X and d(x,y) = 0 if and only if $x = y$.

b. d(x, y) = d(y,x) for all $x, y \in$X

c. d$(x, y) \leq$ d(x, z) + d(y, z) for all $x, y, z \in$X.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

2.4 Definition: (L.G. Haung and X. Zhang [3])

Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in$X.

i. $\{x_n\}$ converges to x if for every c∈E with 0 << c, there is an n_o such that

for all n > n_o, d(x_n,x) << c. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$

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ii. if for any c ∈P with 0 << c, there is an n_o such that for all n, m > n_o,

d(x_n, x_m) << c, then {x_n} is called a Cauchy sequence in X.

iii. (X, d) is called a complete cone metric space, if every Cauchy sequence in X is convergent in X.

2.5 Definition: (H. Lakzian and F. Arabyani [7])

Let X be a cone metric space with metric d. Then a mapping ω: X×X→ E is called ω - distance on X if the following conditions are satisfied

i) $0 \leq \omega(x, y)$ for all $x, y \in$ X;

ii) $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$ for all $x, y, z \in$ X;

iii) if $x_n \to x$ then $\omega(y,x_n) \to \omega(y,x)$ and $\omega(x_n, y) \to \omega(x,y)$

iv) for any 0 << α , there exists 0 << β such that ω(z, x) << β and ω(z, y) << β imply

d(x, y) << α for all α, β ∈ E … (2.5.1)

2.6 Definition: (H. Lakzian and F. Arabyani [7])

Let X be a cone metric space with metric d, let ω be a ω – distance on X, $x \in$ X and { x_n }a sequence in X. Then {x_n} is called a ω- Cauchy sequence whenever for every α ∈ E , 0 << α, there is a positive integer N such that, for all m,n ≥ N , ω(x_m,x_n) << α.

A sequence {x_n} in X is called ω – convergent to a point $x \in$ X whenever for every α ∈ E, 0 << α, there is a positive integer N such that for all n ≥ N , ω(x,x_n) << α.

(X,d) is a complete cone metric space with ω- distance if every Cauchy Sequence is ω – Convergent.

2.7 Lemma : (H. Lakzian, F.Arabyani[7], lemma 2.5)

Let (X,d) be a cone metric space , let p be w - distance on X and let f be a function from X into

E such that $0 \leq f(x)$, for any $x \in$ X. Then a function q from X × X into E given by q(x,y) = f(x) + p(x,y) for each (x , y) ∈ X × X is also a w-distance.

2.8 Example : (H.Lakzian and F.Arabyani[7])

(i) Let (X, d) be a metric space. Then $p = d$ is a w-distance on X.

(ii) Let X be a norm linear space with Euclidean norm.

Then the mapping $p: X \times X \to [0, \infty)$ defined by $p(x, y) = ||x|| + ||y||$ for all $x, y \in$ X is a w- distance on X.

(iii) Let X be a norm linear space with Euclidean norm. Then the mapping

$p: X \times X \to [0, \infty)$ defined by $p(x, y) = ||y||$ for all $x, y \in$ X is a w - distance on X.

(iv) Let X be a cone metric space with metric d, p be a w–distance on X and f bea function from X into E such that $0 \leq f(x)$for any $x \in$X. Then a function

$q: X \times X \to E$ given by $q(x, y) = f(x) + p(x, y)$ for each $(x, y) \in X \times X$ is also a w–distance.

2.9 Definition: (G. Jungck.and B.E. Rhoades [5])

Let S and T be self mappings of a set X. If u = Sx = Tx for some $x \in$ X , then x is called a coincidence point of S and T and u is called a point of coincidence of S and T.

2.10 Definition: (G. Jungck.and B.E. Rhoades [5])

Two self mappings S and T of a set X are said to be weakly compatible if they commute at their coincidence point. i.e; if Su = Tu for some $u \in$ X, then STu = TSu.

2.11 Proposition: (M.Abbas and G. Jungck [1])

Let S and T be weakly compatible self mappings of a set X. If S and T have a unique point of

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coincidence , i.e; u = Sx = Tx, then u is the unique common fixed point of S and T.

2.12 Property: Let (X,d) be a cone metric space. If $\{x_n\}$, $\{y_n\}$ are sequences in X and $x_n \to x$ and $y_n \to y$ then $d(x_n,y_n) \to d(x,y)$.

2.13 Assumption :

$x_n \to x$ (ie; $d(x_n,x) \to 0$)and $x_n \leq y$ implies $x \leq y$

3 Main Results

In this paper , we extend results of Sami Ullah Khan and Arjamand Bano [9] and prove some common fixed point theorem for a pair of weakly compatible mappings in cone metric spaces using ω – distance on X without using normality in cone metric space. Also we extend the results of H.Lakzian and F.Arabyani [7] and 'infimum' of a set is used in H.Lakzian and F.Arabyani [7] which may not be meaningful in the context of a cone metric space.we successfully avoided this difficulty in this paper by obtaining suitable modifications.

3.1 Lemma : Let X be a cone metric space with metric d and $p : X \times X \to E$ is w-distance on Xsatisfies $p(Tx, T^2x) \leq rp(x, Tx) \forall r \in [0,1)$.Then $\{x_n\}$ is a Cauchy sequence.

3.2 Theorem : Let (X, d)be a cone metric space with w - distance p on X and $T: X \to X$.

Suppose that there exists $r \in [0, 1)$such that (i)$p(Tx, T^2x) \leq rp(x, Tx)$, for every $x \in X$ and (ii) $if y \neq Ty$, then there exists $s_y > 0$ such that

$0 < s_y \leq \{p(x, y) + p(x, Tx) \ / \ x \in X\} > 0.$

Then there is a $z \in$ Xsuch that $z = Tz$. Further if $v = T(v)$, then $p(v, v) = 0$.

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Proof : Let $u \in$ Xand define $u_n = T^n u$for any $n \in N$. Then we have, for any $n \in N$,$p(u_n, u_{n+1}) \leq rp(u_{n-1}, u_n) \leq \cdots \leq r^n p(u_0, u_1)$, $0 < r < 1$

So if $m > n$,

$$p(u_n, u_m) \leq p(u_n, u_{n+1}) + \cdots + p(u_{m-1}, u_m)$$
$$\leq r^n p(u_0, u_1) + \ \ldots + r^{m-1} p(u_0, u_1)$$
$$\leq \frac{r^n}{1-r} p(u_0, u_1)$$

Now let $\alpha \in E$with $0 \ll \alpha$be given, then choose $y \in E$with $0 \ll y$such that

$\alpha + N_y(0) \subseteq P$where $N_y(0) = \{z \in X / ||z|| < y\}$.

Also choose a natural numberN_1 such that

$\frac{r^n}{1-r} p(u_0, u_1) \in N_{\frac{\varepsilon}{2}}(0)$then $\frac{r^n}{1-r} p(u_0, u_1) \ll \alpha$

$$\Rightarrow \alpha - \frac{r^n}{1-r} p(u_0, u_1) \in intp$$

$$\therefore p(u_n, u_m) \leq \frac{r^n}{1-r} p(u_0, u_1) \ll \alpha \forall n > m.$$

$\therefore \{u_n\}$ is a Cauchy sequence in X (by Lemma 3.1)

Since X is complete,$\{u_n\}$converges to some point $z \in$ X.

Let $n_0 \in$ Nbe fixed.

Then since $\{u_m\}$converges to z and$p(u_{n_0}, .)$ is continuous,

we have $p(u_{n_0}, z) \leq lim_{m\to\infty} p(u_{n_0}, u_m)$

$$\leq \frac{r^{n_0}}{1-r} p(u_0, u_1)$$

Assume that $z \neq Tz$.Then by hypothesis we have,

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$0 < s_z \leq \{p(x,y) + p(x,Tx) \,/\, x \in X\}$
$0 < s_z \leq \{ p(u_{n_0},z) + p(u_{n_0},u_{n_0+1})/n_0 \in N\}$
$\leq \{ \left(\frac{r^{n_0}}{1-r}\right) p(u_0,u_1) + r^{n_0}p(u_0,u_1) \} \ \ \forall n_0 \in N.$
$\to 0$ as $n_0 \to \infty$
$\Rightarrow \ 0 < s_z \leq 0$

This is a contradiction.

$$\therefore z = Tz$$

$If\, v = Tv$ we have $p(v,v) \quad = p(Tv,T^2v) \leq rp(v,Tv)$
$= rp(v,v).$
$\therefore p(v,v) \leq rp(v,v)$
$\Rightarrow (1-r)p(v,v) \leq 0$
$\Rightarrow p(v,v) = 0.$

3.3 Example : Let (X, d) be a complete cone metric space with w-distance p on X and the mapping $T : X \to X$. Suppose that there exists $r \in [0, 1)$ such that

$p(Tx,T^2x) \leq rp(x,Tx)$ for every $x \in X$ and that $y \neq Ty$. Then there exists $s_y > 0$ such that

$0 < s_y = y - y^2 \leq d(x,y) + d(x,x^2)$ for every $x \in [0, ½]$ and $y \in (0,1)$ and $d(x,y) =$

$|x-y| \ni T(x) = x^2$ on $X = [\,0, ½]$, $r =$ ¾

Then there is a $z \in X$ such that $z = Tz$. Further if $v = T(v)$, then $p(v,v) = 0$.

Proof : Let X = [0, ½] , d and p be usual distances ,

Define T : X → X as T$x = x^2$

Put r = ¾ then

$d(Tx, T^2x) \leq |x^2 - x^4|$
$= x^2(1-x^2)$
$= x^2(1+x)(1-x)$

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$= x(1+x)\,x(1-x)$

$\leq \tfrac{3}{4}(x - x^2)$

$= \tfrac{3}{4}\, d(Tx, T^2x)$

$\therefore d(Tx, T^2x) \leq r\, d(Tx, T^2x) \;\; \forall x \in X.$

3.4 Lemma :

Let $\{y_n\}$ be a sequence in X such that

$$\omega(y_{n+1}, y_n) \leq \lambda\, \omega(y_n, y_{n-1}) \qquad \ldots(3.4.1)$$

where $0 < \lambda < 1$ then $\{y_n\}$ is a Cauchy Sequence in X. Further, if $y_n \to y$ as $n \to \infty$, then

$\omega(y, y) = 0$

Proof : We have by (3.4.1),

$\omega(y_{n+1}, y_n) \leq \lambda^n w(y_1, y_0)\,, \qquad n = 1,2,3,\ldots$

$\therefore$ for $n > m$,

$$\omega(y_n, y_m) \leq \omega(y_n, y_{n-1}) + \omega(y_{n-1}, y_{n-2}) + \omega(y_{n-2}, y_{n-3}) + \ldots + \omega(y_{m+1}, y_m)$$

$$\leq \lambda^{n-1}\omega(y_1, y_0) + \lambda^{n-2}\omega(y_1, y_0) + \ldots + \lambda^{m}\omega(y_1, y_0)$$

$$= [\lambda^{n-1} + \lambda^{n-2} + \lambda^{n-3} + \cdots + \lambda^{m}]\; \omega(y_1, y_0)$$

$$= \left(\frac{\lambda^m}{1-\lambda}\right) \omega(y_1, y_0) \qquad \ldots(3.4.2)$$

$$\to 0 \text{ as } m \to \infty$$

Now let $0 \ll \eta$, choose δ according to (2.5.1).

By (3.4.2), $\omega(y_n, y_m) \ll \delta$ for m large

Hence $\omega(y_{n+1}, y_m) \ll \delta$ and $\omega(y_{n+1}, y_n) \ll \delta$

Hence $d(y_m, y_n) \ll \eta$ by (2.5.1)

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Therefore $\{y_n\}$ is a Cauchy Sequence in (X,d)

Since $\omega(y_{n+1}, y_n) \leq \lambda^n \omega(y_1, y_0)$, n = 1,2,…

we get $\omega(y_{m+k}, y_m) \leq \lambda^m \omega(y_1, y_0)$ …(3.4.3)

for large m and all k

Hence it converges to y , say.

In (3.4.3) , letting $k \to \infty$ we get

$\omega(y , y_m) \leq \lambda^m \omega(y_1, y_0)$

Now letting $m \to \infty$, $\omega(y , y_m) \to 0$

Hence $\omega(y , y) = 0$

3.5 Lemma:

If $w(x , y) = 0$ and $w(y , x) = 0$ then

(i) $w(x , x) = w(y , y) = 0$ and

(ii) $d(x,y) = 0$ so that $x = y$

Proof : Since $\omega(x , x) \leq \omega(x , y) + \omega(y , x) = 0$

we get $\omega(x , x) = 0$. Similarly $\omega(y , y) = 0$

Also we have $\omega(x , y) = 0$ so that $d(x,y) = 0$

Hence $x = y$

3.6 Theorem:

Let (X,d) be a complete cone metric space with ω – distance ω. Let P be a normal cone with normal constant K on X. Suppose that the mappings $S,T : X \to X$ satisfy the following conditions :

(i) The range of T contains the range of S and T(X) is a totally ordered closed subspace of X.

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(ii) $\omega(Sx,Sy) \le r\,[\,\omega(Sx,Ty)+\omega(Sy,Tx)+\omega(Sx,Tx)+\omega(Sy,Ty)+\max\{\omega(Tx,Ty),\omega(Ty,Tx)\}]$ …(3.6.1)

where $r \in [0, 1/7)$ is a constant. Then S and T have a unique coincidence point in X.

Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

Proof : Let $x_0 \in X$. Since S(X) is contained in T(X), we choose a point x_1 in X such that $S(x_0) = T(x_1)$. Continuing this process we choose x_n and x_{n+1} in X such that $S(x_n) = T(x_{n+1})$.

Then $\omega(Tx_{n+1}, Tx_n) = \omega(Sx_n, Sx_{n-1})$

$\le r\,[\,\omega(Sx_n,Tx_{n-1})+\omega(Sx_{n-1},Tx_n)+\omega(Sx_n,Tx_n)+\omega(Sx_{n-1},Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}]$

$= r\,[\,\omega(Tx_{n+1},Tx_{n-1})+\omega(Tx_n,Tx_n)+\omega(Tx_{n+1},Tx_n)+\omega(Tx_n,Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}]$

$\therefore\ w(Tx_{n+1}, Tx_n)$

$\le r\,[\,\omega(Tx_{n+1},Tx_{n-1})+\omega(Tx_n,Tx_n)+\omega(Tx_{n+1},Tx_n)+\omega(Tx_n,Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}]$

Similarly $\omega(Tx_n, Tx_{n+1})$

$\le r\,[\,\omega(Tx_{n+1},Tx_{n-1})+\omega(Tx_n,Tx_n)+\omega(Tx_{n+1},Tx_n)+\omega(Tx_n,Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}]$

$\therefore \max\{\,\omega(Tx_{n+1}, Tx_n), \omega(Tx_{n+1}, Tx_n)\,\}$

$\le r\,[\omega(Tx_{n+1},Tx_{n-1})+\omega(Tx_n,Tx_n)+\omega(Tx_{n+1},Tx_n)+\omega(Tx_n,Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}]$

$\le r\,[\omega(Tx_{n+1},Tx_n) +\omega(Tx_n,Tx_{n-1}) + \omega(Tx_n,Tx_{n-1}) + \omega(Tx_{n-1},Tx_n) + \omega(Tx_{n+1},Tx_n)+$

$\omega(Tx_n,Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}]$

Hence $\alpha_{n+1} \le r\,[\,\alpha_{n+1} + \alpha_n + \alpha_n + \alpha_n + \alpha_{n+1} + \alpha_n + \alpha_n\,]$, where $\alpha_n = \max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}$

$= r\,[\,2\,\alpha_{n+1} + 5\,\alpha_n\,]$

$\therefore\quad (1-2r)\,\alpha_{n+1} \le 5\,r\,\alpha_n$

$\therefore\quad \alpha_{n+1} \le \left(\frac{5r}{1-2r}\right)\alpha_n$

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$\therefore \quad \alpha_n \rightarrow 0$ (since $\frac{5r}{1-2r} < 1$)

Hence { Tx_n } is a Cauchy Sequence in (X,d).

Since T(X) is a totally closed subspace of X, there exists q in T(X) such that $Tx_n \rightarrow q$ as $n \rightarrow \infty$.

Cosequently we can find h in X such that $T(h) = q$. Thus

$\omega(Tx_n, Sh) = \omega(Sx_{n-1}, Sh)$

$\leq r\,[\,\omega(Sx_{n-1},Th)+\omega(Sh,Tx_{n-1})+\omega(Sx_{n-1},Tx_{n-1})+\omega(Sh,Th)+\max\{\omega(Tx_{n-1},Th),\omega(Th,Tx_{n-1})\}]$

$= r\,[\,\omega(Tx_n,Th)+\omega w(Sh,Tx_{n-1})+\omega(Tx_n,Tx_{n-1})+\omega(Sh,Th)+\max\{\omega(Tx_{n-1},Th),\omega(Th,Tx_{n-1})\}]$

On letting $n \rightarrow \infty$

$\omega(Th, Sh) \leq r\,[\,\omega(Th,Th)+\omega(Sh,Th)+\omega(Th,Th)+\omega(Sh,Th)+\max\{\omega(Th,Th),\omega(Th,Th)\}]$

$= 2r\,\omega(Sh,Th)$

$\therefore \ \omega(Th, Sh) \leq 2r\,\omega(Sh,Th)$.

Similarly$\omega(Sh, Th) \leq 2r[\,\omega(Sh,Th)\,]$

$\therefore \ \omega(Th, Sh) = 0$ and $\omega(Sh,Th) = 0$

$\therefore \ Sh = Th$ (by Lemma 3.4)

Hence h is a coincidence point of S and T

Uniqueness : Suppose that there exists a point u in X such that $Su = Tu$

So we have $\omega(Tu,Th) = \omega(Su,Sh)$

$\leq r\,[\,\omega(Su,Th)+\omega(Sh,Tu)+\omega(Su,Tu)+\omega(Sh,Th)+\max\{\omega(Tu,Th),\omega(Th,Tu)\}]$

$= r\,[\,\omega(Tu,Th)+\omega(Th,Tu)+\omega(Tu,Tu)+\omega(Th,Th)+\max\{\omega(Tu,Th),\omega(Th,Tu)\}]$

$\therefore \omega(Tu,Th) \leq r\,[\,\omega(Tu,Th) + \omega(Th,Tu) + \max\{\omega(Tu,Th),\omega(Th,Tu)\}]$

Similarly $\omega(Th,Tu) \leq r\,[\,\omega(Th,Tu) + \omega(Tu,Th) + \max\{\omega(Th,Tu),\omega(Tu,Th)\}]$

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$\therefore$ max {ω(Tu,Th),ω(Th,Tu)} ≤ r [ω(Th,Tu) + ω(Tu,Th) + max{ω(Th,Tu),ω(Tu,Th)}]

Suppose ω(Tu,Th) ≤ r [ω(Th,Tu) + ω(Tu,Th) + ω(Tu,Th)}]

so that (1-2r)ω(Tu,Th) ≤ r ω(Th,Tu)

$\therefore$ ω(Tu,Th) $\leq \frac{r}{1-2r}$ ω(Th,Tu)

Similarly ω(Th,Tu) ≤ $\frac{r}{1-2r}$ ω(Tu,Th)

$\leq \frac{r^2}{(1-2r)^2}$ ω(Th,Tu)

$\therefore$ ω($\mathrm{T}h, \mathrm{T}u$) $>$ ω(Th,Tu) (since $\frac{r}{1-2r}$> 1) , a contradiction

$\therefore$ ω($\mathrm{T}h, \mathrm{T}u$) = 0

Similarly ω($\mathrm{T}u, \mathrm{T}h$) >> ω(Tu,Th) , a contradiction

$\therefore$ω($\mathrm{T}u, \mathrm{T}h$) $= 0$

$\therefore$ Tu = Th

Hence h is a unique coincidence point of S and T.

Now suppose that S and T are weakly compatible.

Then, by Proposition 2.11**,** h is the unique common fixed point of S and T.

Assuming that T(X) is totally ordered , the following result of Sami Ullah Khan and Arjamand Bano [9] follows as a Corollary.

3.7 Corollary: (Theorem 3.2 **,** Sami Ullah Khan and Arjamand Bano [9])

Let (X,d) be a complete cone metric space with ω – distance ω. Let P be a normal cone with normal constant K on X. Suppose that the mappings S,T : X → X satisfy the following conditions :

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(i) The range of T contains the range of S and T(X) is a totally ordered closed subspace of X.

(ii) $\omega(Sx,Sy) \leq r\,[\,\omega(Sx,Ty) + \omega(Sy,Tx) + \omega(Sx,Tx) + \omega(Sy,Ty) + \omega(Tx,Ty)\,]$ …(3.7.1)

where $r \in [0, {}^{1}/_{7}]$ is a constant. Then S and T have a unique coincidence point in X.

Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

3.8 Remark:

In theorem 3.6, if $T = I_X$, the identity map on X , then as a consequence of theorem 3.6, we obtain the following result.

3.9 Corollary:

Let (X,d) be a complete cone metric space with ω – distance ω. Let P be a normal cone with normal constant K on X. Suppose that the mappings $S,T : X \rightarrow X$ satisfy the following conditions :

(i) The range of T contains the range of S and T(X) is a totally ordered closed subspace of X.

(ii) $\omega(Sx,Sy) \leq r\,[\,\omega(Sx,y)+\omega(Sy,x)+\omega(Sx,x)+\omega(Sy,y)+\max\{\omega(x,y),\omega(y,x)\}]$ …(3.9.1)

where $r \in [0, {}^{1}/_{7})$ is a constant. Then S and T have a unique coincidence point in X.

Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

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